Eva on construction of $MU_{\mathbb{R}}$ and its slice differentials

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Recall Adams SS $E_2 = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s} S_2^\wedge \to \pi_{*-s}\Omega$

$$h_j^2 \longrightarrow \theta_j \longrightarrow \neq 0$$

$$\boxed{MU \rightsquigarrow MU_{\mathbb{R}} \rightsquigarrow MU^{((C_8))}} \xrightarrow[\text{localize}]{} D^{-1}MU^{((C_8))} \xrightarrow[\text{fixed pt}]{} \Omega$$

$MU_{\mathbb{R}}$ is a C_2 spectrum, i.e. a suitable functor

$$V \longmapsto ?$$

$$n\rho = \mathbb{C}^n \longmapsto MU(n)$$

$BU(n)$ classifies $\mathbb{C}^n$-vector bundles

C_2 acts on everything in sight

What about other V.

$$\begin{tikzcd} \text{subcat } \{\mathbb{C}^n\} = \mathcal{J}_G' \arrow[r] \arrow[d] & \mathrm{Top}_G \\ \mathcal{J}_G \arrow[ur, dashed, "\text{Kan extension}"'] & \end{tikzcd}$$

We need to replace MU by cofibrant replacement

$$MU_{\mathbb{R}} \simeq \mathrm{hocolim}\ S^{-n\rho_2} \wedge MU(n)$$

- $\Phi^{C_2} MU_{\mathbb{R}} \sim \mathrm{hocolim}\ S^{-n} \wedge MO(n) \sim MO$
- $MU_{\mathbb{R}}$ is a comm ring spectrum $\qquad G = C_8$

$$MU^{((G))} := N_{C_2}^{C_8} MU_{\mathbb{R}} \quad \text{so} \quad \Phi^G MU^{((G))} = MO$$

Want to describe elts in $\pi_* MU^{((G))}$ and compute a slice differential.

Elts in $\pi_* MU^{((G))}$

All is 2-local

<u>Thm</u> $\exists\, M_i \in \pi^u_{2i} MU^{((G))}$ s.t.

- they "generate" $\pi_*^u[MU^{((G))}] = \mathbb{Z}_{(2)}[\gamma^j M_i : \begin{smallmatrix} i \geq 0 \\ 0 \leq j < 3 \end{smallmatrix}]$
- have good fixed point properties

<u>Thm</u>
$$\bigoplus_{j \geq 0} \pi^{C_2}_{j\rho_2} MU_{\mathbb{R}} \cong \bigoplus_{j \geq 0} \pi^u_{2j} MU_{\mathbb{R}}$$

$$\bigoplus_{j \geq 0} \pi^{C_2}_{j\rho_2} MU^{\wedge u}_{\mathbb{R}} \cong \bigoplus_{j \geq 0} \pi^u_{2j} MU^{\wedge u}_{\mathbb{R}}$$

Recall $MU^{((G))} \underset{C_2}{\simeq} MU^{\wedge 4}_{\mathbb{R}}$

$$\bigoplus_{j \geq 0} \pi^{C_2}_{j\rho_2} i^{C_8}_{C_2} MU^{((G))} \cong \bigoplus_{j \geq 0} \pi^u_{2j} i^{C_8}_{C_2} MU^{((G))}$$

$$\bar{M}_i \longleftarrow M_i$$

$N_2^8 \bar{M}_i \in \pi_{i\rho_G} MU^{((G))}$

defined as follows

(internal norm)

$$[S^{i\rho_2}, i_2^8 MU^{((G))}]$$
$$\downarrow N_2^8$$
$$[S^{i\rho_8}, N_2^8 i_2^8 MU^{((G))}]$$
$$\cong$$
$$[S^{i\rho_8}, (MU^{((G))})^{\wedge 4}]$$
$$\downarrow \text{counit}$$
$$[S^{i\rho_8}, MU^{((G))}]$$

$\pi_* MO \cong \mathbb{Z}/2\,[h_i : i \neq 2^k - 1]$

$N\bar{\mu}_1 \longmapsto h_i$ (or 0 if $i = 2^k - 1$)

Alice SS review

$$\begin{tikzcd} & P^1 X \arrow[d] & P_1 X \arrow[l] \\ X \arrow[r] & P^0 X \arrow[d] & \\ & P^{-1} X \arrow[d] & \\ & \vdots & \end{tikzcd}$$

Can get $RO(G)$ SS

$E_2^{s,V} = \pi^G_{V-s} P^{|V|}_{|V|} X \Rightarrow \pi^G_{V-s} X$

Fix $V = * - 2k\sigma$ integer

In general P^n_n is hard but $\bigvee_{k \in \mathbb{Z}} P^k_k$ (wedge of regular slice k-cells)

$\simeq H\mathbb{Z} \wedge$ (some slice cells)

Alice theorem

$\exists$ map $f : V$ slice cells $\longrightarrow MU^{((G))}$

which is an equivalence on P^n_n

$S^0[S^V] := \bigvee_n (S^V)^{\wedge n}$ if $S^V \xrightarrow{x} S^0[S^V]$

$S^0[x] := S[S^V]$

Our wedge of slice cells is

$S^0[\gamma^j \bar{\mu}_i : i \geq 1, 0 \leq j \leq 3]$

$H\mathbb{Z} \wedge S^0[\gamma^j \mu_i] \xrightarrow{\simeq} \bigvee P^n_n MU^{((G))}$

Elements in E_2 $\quad S^{*+*\sigma} \longrightarrow HZ \wedge S^0[G\cdot M_i]$

$$\downarrow \simeq$$

$$\vee\ P^n_n\ MU((G))$$

- $a = a_\sigma : S^0 \longrightarrow S^\sigma \longrightarrow S^\sigma \wedge HZ$

  $a_\sigma \in \pi_{-\sigma} HZ$

- $u = u_{2\sigma}$: For any V, $\mathrm{Res}\ H^G_{|V|}(S^V, \underline{Z}) \xrightarrow{\simeq} H^{\mathcal{U}}_{|V|}(S^V; Z)$

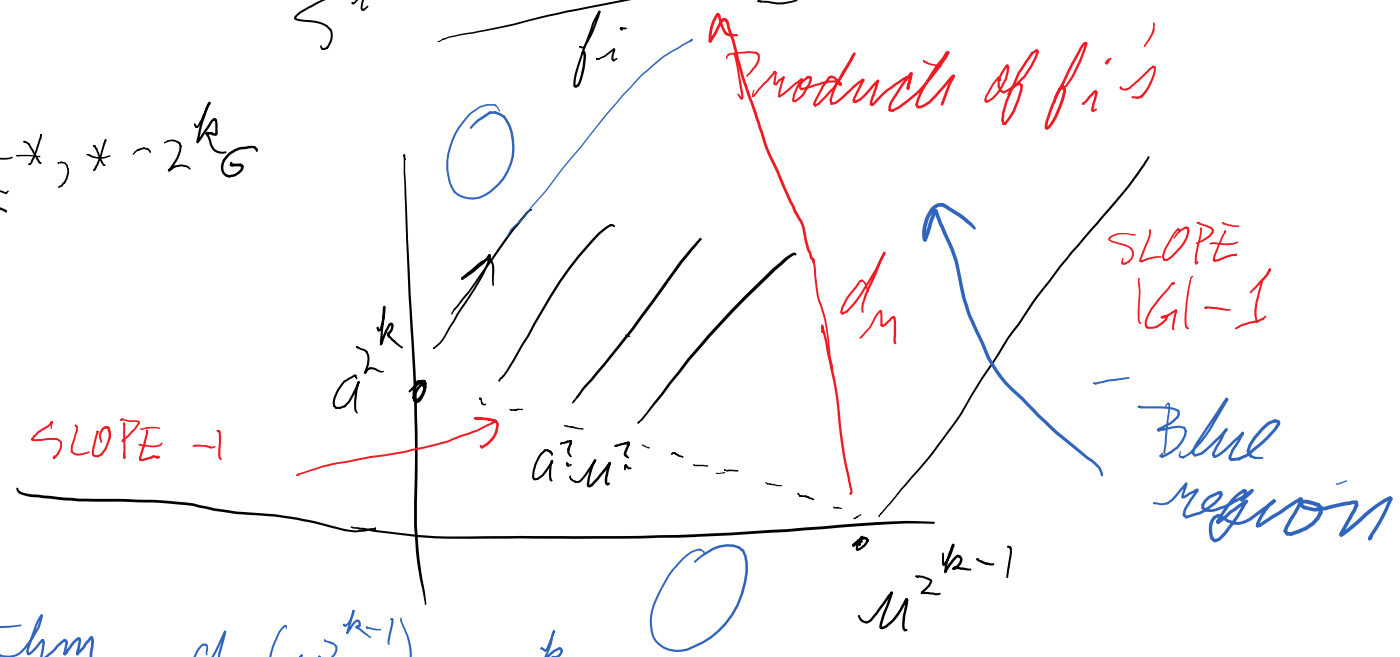
  for V is orientable

  ($2V$ is orientable for any V, e.g. 2σ is orientable)

  $u_{2\sigma} \in \pi_{2-2\sigma} HZ$

- $$\begin{tikzcd} S^{i\rho_G} \arrow[r, "N\bar{M}_i"] & S^0[G\cdot\bar{M}_i] \arrow[r] \arrow[d] & MU((G)) \\ S^i \arrow[u, "a_{i\bar{\rho}_G}"] \arrow[r, "f_i"] & HZ \wedge S^0[G\cdot\bar{M}_i] & \end{tikzcd}$$

$E^{*,*} \sim 2^k G$



Thm $\quad d_n(u^{2^k-1}) = a^{2^k} f_{2^k-1}$

Sketch of proof

a) All elements in blue region are described $\left\{a^{2^k-2l} u_{2l}^{l}\right\}_{0\le l\le 2^{k-1}} \otimes P(f_i)$

b) inverting a leads us to MO, so

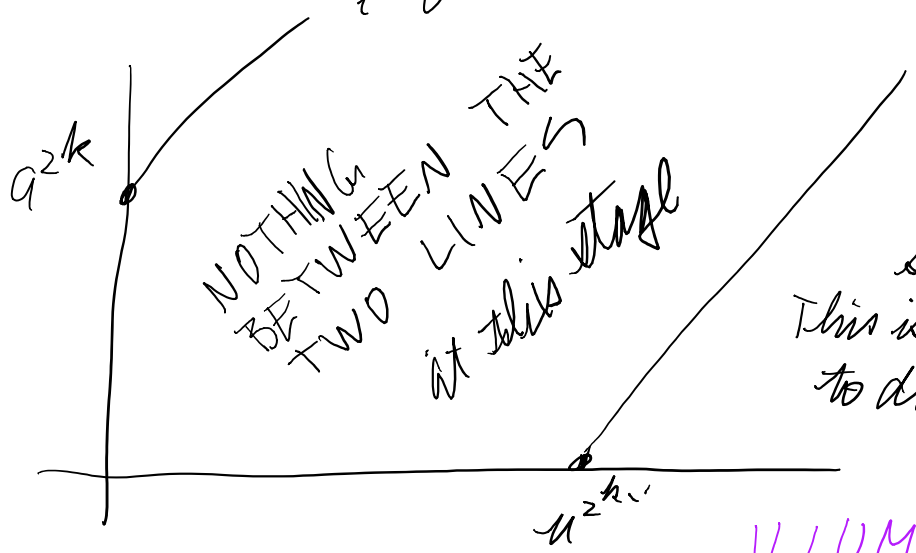
$$\pi_* \Phi^{G_n} X \cong a^{-1}\pi^{G_n} X$$

since $f_{2^k-1} \to h_{2^k-1} = 0$ in $\pi_* MO$

c) $a^? f_{2^k-1}$ must die,

d) It has to be $a^{2^k} f_{2^k-1}$ by calculation

e) Source must be u^{2^k-1}. Use induction on k. Can assume u^l gone for $l < 2^{k-1}$. The f_i and a are permanent cycles so $a^{2^k-2l} u_l f^?$ are gone



$a^{2^k} f_{2^k-1}$ is still here. This is its last chance to die.

ILLUMINATING!

$$a_\sigma \in E_2^{1,1-\sigma} = \pi_{-\sigma} P_0^0 MU((G))$$

$$b_i \in E_2^{(g-1)i,gi} = \pi_i P_{gi}^{gi} MU((G))$$

$$u \in E_2^{0,2-2\sigma} = \pi_{2-2\sigma} P_0^0 MU((G))$$